Application of Oswatitsch's Theorem to Supercritical Airfoil Drag Calculation

G. R. Inger*
Iowa State University, Ames, Iowa 50011

Introduction

THE determination of the local shock wave drag that occurs on a wing section operating at supercritical flight speeds is obviously an important practical aspect of its compressible aerodynamics. It is particularly desirable to have prediction methods for this highly Mach number-sensitive drag that are fundamentally based, not only to provide physical insight but also as a sound basis for experimental data analysis and correlation.

Notwithstanding the development of powerful CFD (Euler/Navier-Stokes) codes for predicting the flow around supercritical airfoils and their associated transonic wave drag rise, where possible, it is helpful to supplement these tools with basic analytical concepts that explain and correlate the trends exhibited by purely numerical predictions, as well as provide scaling laws that underlie experimental data. This note illustrates a case in point: the application of Oswatitsch's classical theorem[1,2] relating wave drag to its integrated entropy rise to explain the nature and behavior of computationally predicted (and experimentally validated) supercritical airfoil C_D rise with Mach number. This is followed by illustrations of its validation by, and use in the correlation of, contemporary experimental data.

Outline of the Theory

Referring to the sketch of a segment dl of a curved shock wave having an overall arc length L shown in Fig. 1, Oswatitsch's theorem is an alternative statement of total momentum conservation that provides the following result for the total shock drag in terms of the integrated entropy rise ${}_1\Delta S_2$ along L:

$$D_{\text{WAVE}} = \frac{T_\infty}{U_\infty} \int_L \rho_1 U_{N_1} ({}_1\Delta S_2)\, dl \qquad (1)$$

where "1" and "2" denote conditions immediately ahead and behind the shock, respectively, and $\rho_1 U_{N_1}$ is the local mass flux across the shock. Murman and Cole[3] have validated this expression against drag values obtained from direct body surface pressure integration. Since this type of integration is sensitive to numerical errors, especially from the nose, tail, and shock wave regions, Eq. (1) offers an alternative drag-determination method that is considerably less error-prone.[4]

Some analysis using the Rankine-Hugioniot shock relations can now be applied to develop a more useful form of Eq. (1) for the wave drag coefficient. Considering the local supersonic flow zone on a supercritical airfoil (Fig. 2) terminated by a slightly curved shock of height h we can write for $L = h$ that

$$\int_L \rho_1 U_{N_1}({}_1\Delta S_2)\, dl = h \cdot \left(\frac{\int_0^h \rho_1 U_{N_1}({}_1\Delta S_2)\, dl}{h} \right)$$

$$= h \cdot [\rho_1 U_{N_1}({}_1\Delta S_2)]_{AV} \qquad (2)$$

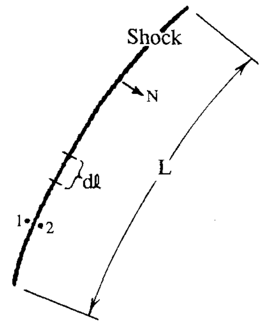

Fig. 1 Segment of curved shock, terminology.

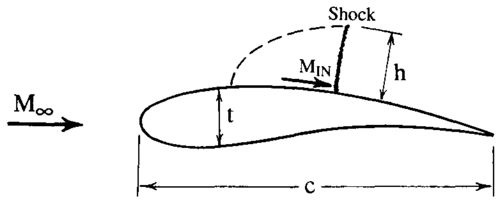

Fig. 2 Local supersonic flow zone on a supercritical airfoil.

where in terms of the gas constant R and specific heat ratio γ, the entropy rise across weak transonic shocks is given by

$$({}_1\Delta S_2) \cong [2\gamma R/3(\gamma + 1)^2](M_{N_1}^2 - 1)^3 \qquad (3)$$

Then introducing the wave drag coefficient based on airfoil chord $C_{D_{\text{WAVE}}} \equiv 2D_{\text{WAVE}}/\rho_\infty U_\infty^2 c$, Eqs. (1–3) yield

$$C_{D_{\text{WAVE}}} = K_1(M_\infty, t/c) \cdot (h/c) \cdot (M_{1_N}^2 - 1)_B^3 \qquad (4a)$$

where the subscript B denotes body conditions at the airfoil surface, t/c is the airfoil thickness ratio, $M_{1_N} > 1$ the preshock Mach number and

$$K_1 \equiv \frac{4}{3(\gamma + 1)^2 M_\infty^2} \frac{[\rho_1 U_{N1}(M_{N_1}^2 - 1)^3]_{AV}}{\rho_\infty U_\infty (M_{N_1}^2 - 1)_B^3} \qquad (4b)$$

is an appropriate freestream Mach number and airfoil shape-dependent factor.

Now, study of experimental data on the local transonic flow over typical airfoils establishes the dependence that

$$(h/c) \cong K_2(M_\infty, t/c) \cdot (M_{1_N}^2 - 1)_B^{1+\varepsilon} \qquad (5)$$

where K_2 and $\varepsilon = \varepsilon(M_\infty, t/c)$ are additional constants with the exponent ε being generally small with respect to unity. Indeed, theory and experiment[5,6] indicate that $\varepsilon \cong 0.3$–0.4 over a range of conditions as illustrated in Fig. 3. Thus, we may finally obtain from Eq. (4a) the expression

$$C_{D_{\text{WAVE}}} \cong K_3(M_\infty, t/c) \cdot (M_{1_N}^2 - 1)_B^{4+\varepsilon} \qquad (6)$$

where $\varepsilon \approx 0.3$–0.4 and K_3 is some constant that contains the remaining (and very weak) integrated effect of the preshock conditions along the shock length.

Applications

Equation (6) provides a fundamentally based expression for scaling the effects of local preshock Mach number on wave drag that proves useful in practical aerodynamic work. It has been verified by numerous experimental measurements and numerical predictions of contemporary supercritical airfoil flowfield computer codes. For example, Fig. 4 illustrates comparisons with several sets of airfoil data[7] in which the wave

Presented as Paper 91-3210 at the AIAA 9th Applied Aerodynamics Conference, Baltimore, MD, Sept. 23–25, 1991; received Oct. 24, 1991; revision received May 1, 1992; accepted for publication May 1, 1992.

*Professor, Department of Aerospace Engineering and Engineering Mechanics. Associate Fellow AIAA.

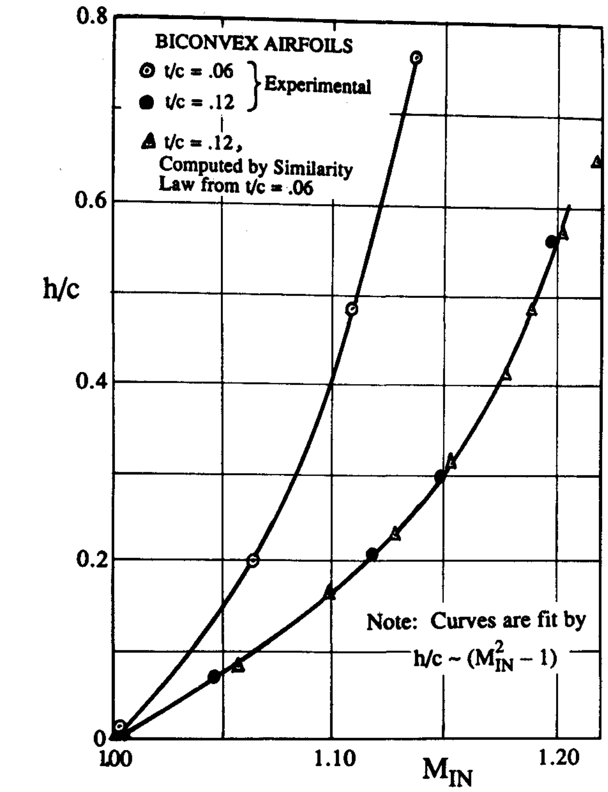

Fig. 3 Variation of terminating shock height with preshock Mach number[6] (after Liepmann, Ashkenas, and Cole: see Ref. 6).

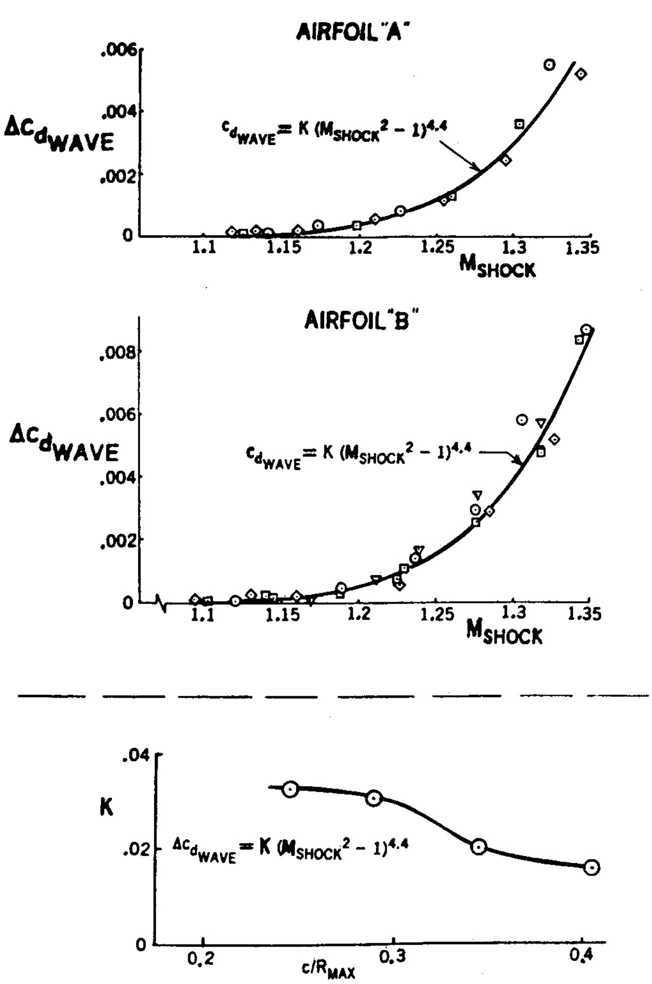

Fig. 4 Correlation of experimental wave drag coefficient with preshock Mach number for several supercritical airfoils.[7]

drag was extracted from measurements of the wake momentum defect; it can be seen that the crucial and very sensitive local Mach number-dependence aspect of expression (6) is well supported by the data.

Another useful scaling-law aspect of Eq. (6) can also be demonstrated for thin airfoils by some additional theoretical development. Specifically, we seek to re-express Eq. (6) in terms of the critical flight Mach number $M_{\infty_{CRIT}}$. Using the isentropic relationship

$$C_p(M) = \frac{2}{\gamma M_\infty^2}\left\{\left[\frac{2 + (\gamma - 1)M_\infty^2}{2 + (\gamma - 1)M^2}\right]^{\gamma/(\gamma-1)} - 1\right\} \tag{7}$$

and then applying the Prandtl-Glauert rule twice, we write

$$Cp^* \equiv Cp(1) = \frac{CP_{\mathrm{INC,MIN}}}{\sqrt{1 - M_{\infty_{CRIT}}^2}} \tag{8a}$$

$$Cp_1 \equiv Cp(M_{1_N}) = \frac{CP_{\mathrm{INC,SHOCK\ LOC}}}{\sqrt{1 - M_\infty^2}} \tag{8b}$$

where $M_\infty = M_{\infty_{CRIT}} + \Delta M$ is slightly higher than $M_{\infty_{CRIT}}$. Then assuming $CP_{\mathrm{INC,SHOCK\ LOC}}$ is approximately the same as $CP_{\mathrm{INC,MIN}}$ and equating their values from Eqs. (8a) and (8b) using Eq. (7), and taking $\Delta M/M_{\infty_{CRIT}}$ to be small compared to unity, some lengthy algebra yields the expression

$$M_{1N}^2 - 1 \approx C\cdot(M_\infty - M_{\infty_{CRIT}}) \tag{9a}$$

where [denoting $\lambda \equiv 2 + (\gamma - 1)M_{\infty_{CRIT}}^2$]

$$C = \frac{2(\gamma + 1)M_{\infty_{CRIT}}}{\lambda} - \frac{(\gamma + 1)(2 - M_{\infty_{CRIT}}^2)}{\gamma M_{\infty_{CRIT}}(1 - M_{\infty_{CRIT}}^2)} \cdot\left[1 - \left(\frac{\gamma + 1}{\lambda}\right)^{\gamma/\gamma - 1}\right] \tag{9b}$$

is a parameter depending on the particular airfoil shape and angle of attack. Substituting Eq. (9a) we see that Eq. (6) yields

$$C_D \sim (M_\infty - M_{\mathrm{CRIT}})^4 = K_4(M_\infty - M_{\mathrm{CRIT}})^4 \tag{10}$$

which in fact agrees with a semiempirical expression fitting transonic drag rise data that was obtained some time ago by Lock[5] (who give $k_4 \cong 20$). A design-oriented application of this result has recently been given by Malone and Mason.[8]

References

[1]Oswatitsch, K., "Der Verdichtungsstoss bei der stationären Umströmung flacher Profile," *ZAMM*, Vol. 29, 1949, pp. 129–141.

[2]Sears, W. R. (ed.), *Foundation of High Speed Aerodynamics*, Princeton University Press, Princeton, NJ, 1952, pp. 23, 24.

[3]Murman, E., and Cole, J. D., "Analysis of Embedded Shock Waves Calculated by Relaxation Methods," *AIAA Journal*, Vol. 12, No. 5, 1974, pp. 626–632.

[4]Cole, J. D., and Cook, L. P., *Transonic Aerodynamics*, North Holland, New York, 1986, pp. 328–348.

[5]Hilton, H. W., *High Speed Aerodynamics*, Longmans, Green and Co., London, 1951, pp. 47–50.

[6]Shapiro, A. M., *Dynamics and Thermodynamics of Compressible Fluid*, Vol. 2, Ronald Press, New York, 1956, pp. 858–860.

[7]Lynch, F. T., Bui, M., and Patel, D., "Some Fundamental Concepts in the Design Analysis and Testing of Transonic Airfoils," private communication, Long Beach, CA, Oct. 1985.

[8]Malone, B., and Mason, W. H., "Aircraft Concept Development Using the Global Sensitivity Approach and Algebraic Technology Models," Virginia Tech. Dept. of Aerospace Engineering Rept. VPI-Aero-184, Blacksburg, VA, Dec. 1991.